Lecture 5: Characteristics in Higher Dimensions & for more general Equations

Higher Dimensions

- First, let us translate the model problem to higher dimensions.

Let $R \subseteq \mathbb{R}^n$ be a bounded region with C^1 boundary. The mass in R may again be computed by integrating the density, $u(t,x)$

$$m(t) = \int_R u(t,x)\,dx$$

The flow of u is given by the vector-valued flux $q(t,x)$ that represents the mass flow: the rate at which mass passes through the $(n-1)$-dimensional surface $V \subseteq R$ is given by $\int_V \eta \cdot q(t,x)\,dS$ (normal vector η).

In particular, the rate at which mass leaves R is $\int_{\partial R} \eta \cdot q(t,x)\,dS$ for $\eta(x)$ the <u>outward</u> unit normal to ∂R at x.

Conservation of mass then gives, as before,

$$\frac{dm}{dt} = -\int_{\partial R} \eta \cdot q\,dS$$

which theorem do we use?

Assuming q is C^1, $\int_{\partial R} \eta \cdot q\,dS = \int_R \nabla_x \cdot q\,dx$

$\hookrightarrow$ divergence in space only.

If u is also C^1, the Leibniz Rule allows $\frac{dm}{dt} = \int_R \frac{\partial u}{\partial t}\,dx$ and

$$\int_R \left(\frac{\partial u}{\partial t} + \nabla \cdot q\right)dx = 0$$

As in one spatial dimension, R is arbitrary and so

$$\frac{\partial u}{\partial t} + \nabla \cdot q = 0$$

Assuming again flow = concentration $\times$ velocity or $q = vu$ for v independent of u

$$0 = \frac{\partial u}{\partial t} + \nabla \cdot (vu) = \frac{\partial u}{\partial t} + (\nabla \cdot v)u + v(\nabla \cdot u)$$

the linear conservation equation in $\mathbb{R}^n$. We then study

$$\text{(F)} \quad \frac{\partial u}{\partial t} + v \cdot (\nabla u) + w = 0$$

he goal of the method of Characteristics is to reduce to one dimension, so we try to apply it similarly. We consider a curve $x(t)$,

Set $\frac{dx}{dt}(t) = v(t, x(t))$ and $x(t_0) = x_0$. We have local existence of a solution by Picard-Lindelöf.

$\rightsquigarrow$ Do the sizes of the vectors match?

· Again, set $\frac{Du}{Dt}(t) = \frac{d}{dt} u(t, x(t))$ and we have

$\boxed{\text{Th}^{\underline{m}}}$ On each characteristic curve, (F) reduces to the ODE.

$$\frac{Du}{Dt} + w(t, x(t), u(x(t))) = 0$$

$\boxed{\text{Pf}}$ $\frac{Du}{Dt}(t) = \frac{\partial u}{\partial t}(t, x(t)) + \nabla u(t, x(t)) \cdot \frac{dx}{dt}(t)$

$= \frac{\partial u}{\partial t}(t, x(t)) + v \cdot \nabla u$ by the assumption for the characteristics.

ex.) Let $U = \mathbb{R} \times \underbrace{[-1,1]}_{\sim \text{a "tube"}}$ and $v(t, x) = (1 - x_2^2, 0)$. Consider $\frac{\partial u}{\partial t} + \nabla \cdot (uv) = 0$

Notice $\nabla \cdot v = 0$ and v vanishes on ∂U (in $\mathbb{R}^2$).

If $\dot{x}(t) = (1 - x_2^2, 0)$ and $x(0) = a, b$

$x(t) = (a + (1 - b^2)t, b)$

Given initial condition $u(0, x) = g(x)$, we may solve

$$\begin{cases} \frac{Du}{Dt} = 0 \\ u(x(0), 0) = g(a, b) \end{cases}$$

↳ typo, swap the orders

to see $u(t, a + (1-b^2)t, b) = g(a, b)$ or

$u(t, x, y) = g(x - (1-y^2)t, y)$

Picture of flow given initial "spot"

- the area stays constant, showing conservation of mass.

Quasilinear Equations

- we now allow some dependence on u for the flux:

$$(G)\quad \frac{\partial u}{\partial t} + a(u)\cdot\nabla u = 0 \qquad \text{for} \quad a(u) = \frac{dq}{du}$$

↳ like velocity above

Quasilinear - linear in the highest-order derivatives

$\boxed{Th^{\underline{m}}}$ Suppose that $u\in C^1([0,T]\times U)$ is a solution to (G) with $a\in C^1(\mathbb{R};\mathbb{R})$. Then, for each $x_0\in U$, u is constant on the characteristic line defined by

$$x(t) = x_0 + a(u(0,x_0))t.$$

$\boxed{Pf}$ Suppose that a solution u exists. Let $x(t)$ solve the ODE

$$\dot{x}(t) = a(u(t,x(t))),\quad x(0)=x_0.$$

Notice $\frac{d}{dt}u(t,x(t)) = \frac{\partial u}{\partial t}(t,x(t)) + \nabla u(t,x(t))\frac{dx}{dt} = 0,$

Hence, $u(t,x(t)) = u(0,x_0)$. This also gives $a(u(t,x(t))) = a(u(0,x_0))$ and so $x(t) = x_0 + a(u(0,x_0))t.$ $\square$

⇝ Unlike the linear case, $x(t)$ depends on $u(0,x_0)$.

⇝ This only holds if we already have a solution. It doesn't necessarily provide a solution.

- Example: Traffic Equation

Let $u(t,x)$ denote the traffic density on a stretch of road at a given time/position. For modeling, assume $u\in C^1$ (approximating the true discrete situation).

Since traffic density affects car velocity, we set a max velocity or speed limit v_m that occurs when $u=0$ and let velocity slow or decrease until we hit a max. density u_m.

i.e. $\quad V(u) = v_m(1 - u/u_m)$

For simplicity, let $v_m = u_m = 1$, so $V(u) = 1-u$. The flux is $q(u) = u-u^2$ giving the traffic equation

$$(H)\quad \frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = 0$$

assume $\quad u(0,x) = h(x)\quad$ for some $h:\mathbb{R}\to[0,1]$.

Let $u(t,x)$ be some solution. The theorem gives

$$x(t) = x_0 + (1-2h(x_0))t$$

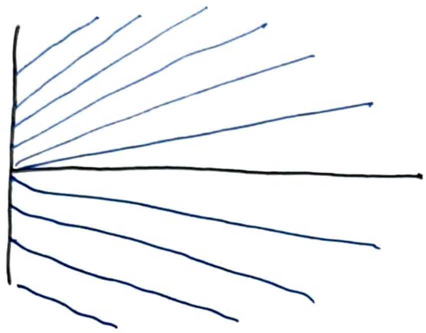
so that

$$(I)\quad u(t, x_0 + (1-2h(x_0))t) = h(x_0)$$

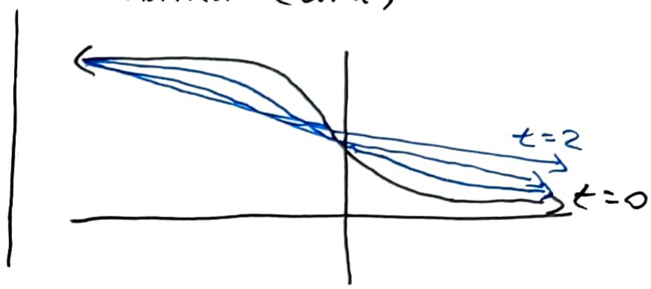
1.) Let $h(x) = \frac{1}{2} - \frac{1}{\pi}\arctan(20x)$

the characteristic lines look like

↳ cars stopped at a light.

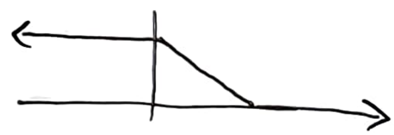


Solution Curves



to find $u(t,x)$, we invert $x = x_0 + (1-2h(x_0))t$
to solve for x_0. This isn't possible to do explicitly, but may be done numerically.

2.) $h(x) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 < x < 1 \\ 0 & x \ge 1 \end{cases}$



↳ not C^1, but we still investigate.

Characteristics

$$x(t) = \begin{cases} x_0 - t & x_0 \le 0 \\ x_0 + (2x_0 - 1)t & 0 < x_0 < 1 \\ x_0 + t & x_0 \ge 1 \end{cases}$$

Solving for x_0

$x = x_0 - t$ if $x_0 \le 0$ $\iff$ $x_0 = x + t$ if $x \le -t$

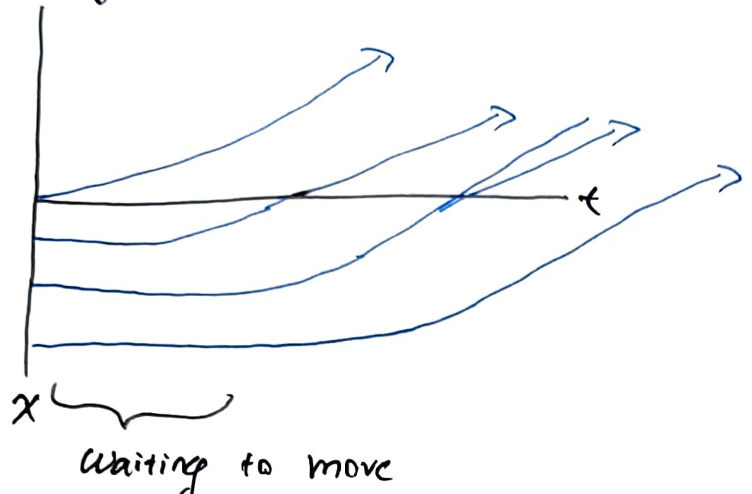
$x = x_0 + 2x_0 t - t$ if $0 < x_0 < 1$ $\iff$ $\frac{x+t}{1+2t} = x_0$ if $-t < x < 1+t$

$x_0 = x + t$ if $x_0 \ge 1$ $\iff$ $x_0 = x - t$ if $x \ge 1+t$

or $u(t,x) = h(x_0) = \begin{cases} 1 & x \le -t \\ 1 - \frac{x+t}{1+2t} & -t < x < 1+t \\ 0 & x \ge 1+t \end{cases}$

- Check that u solves (G) except on lines $x = -t$, $x = 1+t$.

• Calculating the velocity gives car trajectories that look like

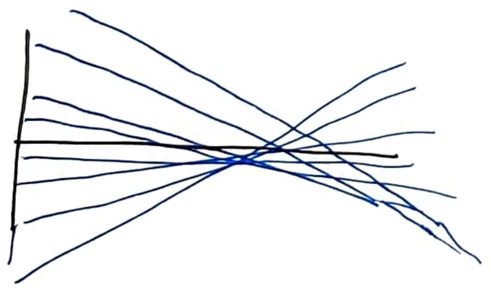


- Cars further from the stop must wait longer, as we'd expect.

3.) A "bad" $h(x)$

$$h(x) = \tfrac{1}{2} + \tfrac{1}{\pi} \arctan(20x)$$

gives characteristics



• Crossing characteristics mean that the solution $u(t,x)$ breaks down (at the crossing points).

• Physically, we would have a traffic jam. This is a "shock" in our system.